

Coherence for vectorial waves and majorization

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We show that majorization provides a powerful approach to the coherence conveyed by partially polarized transversal electromagnetic waves. Here we present the formalism, provide some examples and compare with standard measures of polarization and coherence of vectorial waves.

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I. INTRODUCTION

Coherence is a basic physical property that emerges in very different contexts, from classical optics to quantum mechanics. Therefore the efforts to find a proper measure of coherence seem justified, specially after its identification as a resource [1].

In this work we focus on the assessment of the coherence conveyed by pairs of partially polarized electromagnetic beams in classical optics. Coherence has two extreme physical manifestations: interference, as coherence when superimposing beams with the same vibrational state, and polarization, as coherence when superimposing beams with orthogonal vibration states. This makes specially attractive the analysis of coherence in the superposition of partially polarized waves. The complexity of the subject has motivated the introduction of complementary approaches focusing on different perspectives [2–4]. Among them, it makes sense to address the idea of global coherence embracing both interference and polarization at once [5, 6]. Parallels can also be drawn to its quantum counterpart through the idea of state purity [7–9].

Besides standard measures, entropies are in general good candidates to assess statistical properties. Entropy has already been used to measure polarization [8, 10], being specially useful for situations beyond Gaussian or second-order statistics. A rather attractive feature when dealing with entropies is the emergence of majorization as a kind of meta-measure that establishes a once-for-all partial ordering of statistics [11]. This partial relation is respected by the Schur-concave functions, that include the Shannon and Rényi entropies [12].

We will study the coherence conveyed by two transversal waves, this is four electric field components. This may be the case of the interference of partially polarized waves emerging from the pinholes of a Young interferometer. The statistics will be fully specified by their second-order field correlations in the space-frequency domain, this is a 4×4 positive cross-spectral density tensor $\mathbf{\Gamma}$. The entropies and majorization will be applied to its normalized spectrum, this is to the eigenvalues of $\mathbf{\Gamma}/\text{tr}\mathbf{\Gamma}$. As a proper precedent we can mention the application of majorization to the analysis of polarization of three-dimensional fields [13]. The conclusions will be contrasted to other accounts of coherence more focused on the ideas of field correlation or fringe visibility.

II. SETTINGS: FIELD STATES AND COHERENCE MEASURES

For definiteness let us focus on two vectorial electric fields \mathbf{E} at two spatial points $\mathbf{r}_{1,2}$ with just two non vanishing components at each point, say $E_{x,y}$. This can be the transverse electric field at the two pinholes of a Young interferometer. The complete system is made of four scalar electric fields that we will consider in the space-frequency domain $E_\ell(\mathbf{r}_j, \omega)$ with $j = 1, 2$, $\ell = x, y$, and the temporal frequency ω will be omitted from now on. Their statistics will be completely accounted for by the second-order field correlations gathered by the cross-spectral density tensor, that in our case is a 4×4 non-negative matrix $\mathbf{\Gamma}$, with

$$\mathbf{\Gamma} = \begin{pmatrix} \Gamma_{1,1} & \Gamma_{1,2} \\ \Gamma_{2,1} & \Gamma_{2,2} \end{pmatrix}, \quad (2.1)$$

where $\Gamma_{j,k}$ are 2×2 correlation matrices with $\Gamma_{j,k} = \Gamma_{k,j}^\dagger$,

$$\Gamma_{j,k} = \langle \mathbf{E}(\mathbf{r}_j) \mathbf{E}^\dagger(\mathbf{r}_k) \rangle, \quad \mathbf{E}(\mathbf{r}_j) = \begin{pmatrix} E_x(\mathbf{r}_j) \\ E_y(\mathbf{r}_j) \end{pmatrix} \quad (2.2)$$

with \dagger representing Hermitian conjugation. Moreover, we can collect the four components into a single four-dimensional vector \mathbf{E} with components $E_1 = E_x(\mathbf{r}_1)$, $E_2 = E_y(\mathbf{r}_1)$, $E_3 = E_x(\mathbf{r}_2)$, $E_4 = E_y(\mathbf{r}_2)$, so that $\Gamma_{j,k} = \langle E_j E_k^* \rangle$, for $j, k = 1, \dots, 4$. Under these conditions every account of coherence and polarization must be a function of these matrices $\mathbf{\Gamma}$ and $\Gamma_{j,k}$.

Scalar interferometric coherence.— For completeness we recall the standard measure of coherence μ in the simplest case of two scalar fields $E_{1,2}$

$$\mu = \frac{\langle E_1 E_2^* \rangle}{\sqrt{\langle |E_1|^2 \rangle \langle |E_2|^2 \rangle}}. \quad (2.3)$$

We refer to this as interferometric coherence in the sense of being the key factor controlling the visibility of the interference fringes obtained when superimposing $E_{1,2}$.

Polarization.— Polarization expresses the coherence between two components, say $E_{x,y}$ of a transverse field vector at a given spacial point as

$$P^2 = 2 \frac{\text{tr}(\mathbf{\Gamma}^2)}{(\text{tr}\mathbf{\Gamma})^2} - 1, \quad (2.4)$$

where here $\mathbf{\Gamma}$ refers to the 2×2 coherency matrix with matrix elements $\Gamma_{j,k} = \langle E_j E_k^* \rangle$, $j, k = x, y$. It is worth noting that this is the maximum interferometric coherence (2.3) that can be reached between any two field components $E_{1,2}$ obtained from $E_{x,y}$ by a unitary 2×2 matrix $U \in U(2)$, this is to say $\mu(U) \leq P$. After computing P for $\mathbf{E}(\mathbf{r}_{1,2})$ the two degrees of polarization $P_{1,2}$ can be combined to provide a single value, for example as in Refs. [14].

Vectorial interferometric coherence.— The interferometric account of coherence μ becomes more complex when the interfering fields are vectorial. Accordingly, different definitions have been proposed as different generalizations of Eq. (2.3), such as [2–4]:

$$\mu_{KW} = \frac{\text{tr} \mathbf{\Gamma}_{1,2}}{\sqrt{\text{tr} \mathbf{\Gamma}_{1,1} \text{tr} \mathbf{\Gamma}_{2,2}}}, \quad (2.5)$$

$$\mu_{TSF}^2 = \frac{\text{tr} (\mathbf{\Gamma}_{1,2} \mathbf{\Gamma}_{1,2}^\dagger)}{\text{tr} \mathbf{\Gamma}_{1,1} \text{tr} \mathbf{\Gamma}_{2,2}}, \quad (2.6)$$

and, when the corresponding inverses exist,

$$\mu_{S,I} = \text{singular values of } \mathbf{\Gamma}_{1,1}^{-1/2} \mathbf{\Gamma}_{1,2} \mathbf{\Gamma}_{2,2}^{-1/2}, \quad (2.7)$$

where $\mu_{S,I}$ are real and $\mu_S \geq \mu_I \geq 0$.

Global coherence.— In this work we are mostly interested in regarding the four field components as a whole, asking for the global coherence conveyed by the complete field. This should comprise both the polarization and interferometric contributions, and should be expressed by the whole $\mathbf{\Gamma}$ instead of its sub-matrices $\mathbf{\Gamma}_{j,k}$.

For another perspective, we may say that polarization and interferometric coherence depends on a choice of field modes, which is the equivalent of the basis dependence in quantum mechanics. Thus we can attempt a mode-independent approach where coherence may be referred to as intrinsic, *per se*, or global [1]. This would be analogous to the role played by the degree of polarization (2.4) versus the degree of coherence (2.3) regarding two scalar electric fields.

A convenient generalization of the degree of polarization (2.4) to four-dimensional fields can be

$$\mu_g^2 = \frac{4}{3} \frac{\text{tr} (\mathbf{\Gamma}^2)}{(\text{tr} \mathbf{\Gamma})^2} - \frac{1}{3} = \frac{4}{3} \lambda^2 - \frac{1}{3}, \quad (2.8)$$

where λ is a four-dimensional vector containing the eigenvalues of $\mathbf{\Gamma}/\text{tr} \mathbf{\Gamma}$ [6]. An alternative assessment along this line can be found in Ref. [5]. In the next section we show that μ_g is actually a case of Rényi entropy.

III. MAJORIZATION AND GLOBAL DEGREE OF COHERENCE

Next we recall the idea of majorization and its relation with coherence measures as functions of the $N \times N$

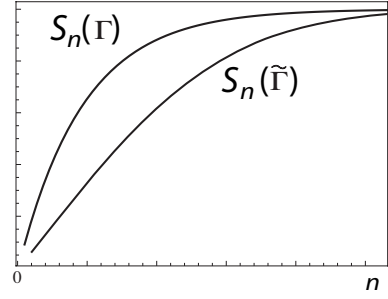


FIG. 1: Relation between partial ordered sums S_n when the majorization $\tilde{\mathbf{\Gamma}} \prec \mathbf{\Gamma}$ holds.

coherency matrix $\mathbf{\Gamma}$. The main idea is that coherence is reflected in the dispersion of the eigenvalues of $\mathbf{\Gamma}/\text{tr} \mathbf{\Gamma}$, that are real, positive, and normalized $\sum_{j=1}^N \lambda_j = 1$. We have two clear extremes. We have full coherence for those $\mathbf{\Gamma}_p$ with only one eigenvalue different from zero, say $\lambda_1 = 1, \lambda_{j \neq 1} = 0$. On the other extreme, there is complete lack of coherence for those $\mathbf{\Gamma}_I$ where all the eigenvalues are equal $\lambda_j = 1/N$, so that $\mathbf{\Gamma}_I$ is proportional to the identity. In between, the degree of coherence may be assessed using many possible functions of λ . Most of them are of entropic nature, such as the Rényi entropies [12]

$$R_q(\lambda) = \frac{1}{1-q} \ln \left(\sum_{j=1}^N \lambda_j^q \right), \quad (3.1)$$

where $q > 0$ is an index labeling different entropies. The limiting case $q \rightarrow 1$ is the Shannon entropy $R_1 = -\sum_{j=1}^N \lambda_j \ln \lambda_j$. For example, we have that μ_g in Eq. (2.8) is essentially R_2 as

$$\mu_g^2 = \frac{4}{3} e^{-R_2} - \frac{1}{3}. \quad (3.2)$$

We say that $\mathbf{\Gamma}$ majorizes $\tilde{\mathbf{\Gamma}}$, which will be expressed as $\tilde{\mathbf{\Gamma}} \prec \mathbf{\Gamma}$, when the following relation between all the ordered partial sums S_n of their corresponding eigenvalues holds,

$$S_n(\tilde{\mathbf{\Gamma}}) = \sum_{j=1}^n \tilde{\lambda}_j^\downarrow \leq \sum_{k=1}^n \lambda_k^\downarrow = S_n(\mathbf{\Gamma}), \quad (3.3)$$

for all $n = 1, 2, \dots, N$, where the superscript \downarrow denotes the same λ_j but arranged in decreasing order

$$\lambda_1^\downarrow \geq \lambda_2^\downarrow \geq \dots \geq \lambda_N^\downarrow. \quad (3.4)$$

Throughout we will say that $\mathbf{\Gamma}$ and $\tilde{\mathbf{\Gamma}}$ are comparable if one majorizes the other.

Next, some interesting facts about majorization. Majorization holds if and only if $\tilde{\lambda} = M\lambda$ where M is a doubly stochastic matrix, so that $\tilde{\mathbf{\Gamma}}$ is more uniform or

more mixed than Γ . For all Γ we have $\Gamma_I \prec \Gamma \prec \Gamma_p$ [15]. Whenever two Γ are comparable, the result is respected by all Schur-concave functions that includes the Rényi entropies: if $\tilde{\Gamma} \prec \Gamma$ then $R_q(\tilde{\Gamma}) > R_q(\Gamma)$ for all q .

After all these facts we may say if $\tilde{\Gamma} \prec \Gamma$ then Γ is more coherent than $\tilde{\Gamma}$. Majorization is a partial ordering relation, so that there are incomparable states: this is neither $\tilde{\Gamma} \prec \Gamma$ nor $\Gamma \prec \tilde{\Gamma}$. In such a case the ordered sums in Fig. 1 will intersect and the entropies will provide contradictory conclusions, such that $R_q(\tilde{\Gamma}) > R_q(\Gamma)$ while $R_p(\tilde{\Gamma}) < R_p(\Gamma)$ for different entropies $p \neq q$.

The case of 2×2 matrices Γ is rather trivial since after normalization the spectrum λ depends on a single parameter. Thus any two Γ are comparable and there is no room for ambiguities nor discrepancies between measures. The case of 3×3 matrices Γ has been completely addressed in a recent work [13] regarding Γ as representing polarization in three dimensions. Thus in this work we address 4×4 matrices Γ , that may be representing the two transversal fields described above. In the next sections we analyze some relevant and illustrative cases.

IV. UNPOLARIZED BEAMS OF THE SAME INTENSITY

In this case by means of suitable $U(2)$ transformations the Γ matrix can be arranged so that

$$\Gamma_{1,1} = \Gamma_{2,2} = I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_{1,2} = I \begin{pmatrix} \mu_S & 0 \\ 0 & \mu_I \end{pmatrix}, \quad (4.1)$$

where I represents the intensity of each component, that will play no role on the following.

The corresponding values of the above measures of polarization and interferometric coherence are: $P_1 = P_2 = 0$,

$$\mu_{KW} = \frac{1}{2}(\mu_S + \mu_I), \quad \mu_{TSF}^2 = \frac{1}{4}(\mu_S^2 + \mu_I^2), \quad (4.2)$$

while the global coherence is

$$\mu_g^2 = \frac{1}{6}(\mu_S^2 + \mu_I^2). \quad (4.3)$$

The properly ordered eigenvalues of $\Gamma/\text{tr}\Gamma$ in decreasing order are

$$\begin{aligned} \lambda_1^\downarrow &= \frac{1}{4}(1 + \mu_S), & \lambda_2^\downarrow &= \frac{1}{4}(1 + \mu_I), \\ \lambda_3^\downarrow &= \frac{1}{4}(1 - \mu_I), & \lambda_4^\downarrow &= \frac{1}{4}(1 - \mu_S), \end{aligned} \quad (4.4)$$

leading to the following ordered partial sums

$$\begin{aligned} S_1 &= \frac{1}{4}(1 + \mu_S), & S_2 &= \frac{1}{4}(2 + \mu_S + \mu_I), \\ S_3 &= \frac{1}{4}(3 + \mu_S), & S_4 &= 1. \end{aligned} \quad (4.5)$$

After these expressions for S_n majorization is actually determined just by the two first conditions in Eq. (3.3) so that $\tilde{\Gamma} \prec \Gamma$ if and only if the following two conditions are satisfied

$$\mu_S \geq \tilde{\mu}_S, \quad \mu_S + \mu_I \geq \tilde{\mu}_S + \tilde{\mu}_I. \quad (4.6)$$

Since the degree of polarization of $\mathbf{E}(\mathbf{r}_j)$ vanishes one might ask whether the majorization conditions (4.6) are equivalent to any of the above degrees of interferometric coherence in Eq. (4.2) or the global coherence in Eq. (4.3). The result is negative. To show this the picture in Fig. 2 may be useful. This is a $\mu_{S,I}$ plane where any Γ is represented by a point in the region of the first quadrant below the bisecting line. This is to respect the condition $\mu_S \geq \mu_I \geq 0$. The dotted line represents the condition $\mu_S \geq \tilde{\mu}_S$, while the dashed line represents the condition $\mu_S + \mu_I \geq \tilde{\mu}_S + \tilde{\mu}_I$. Therefore relations (4.6) define three regions:

α : Points $\tilde{\Gamma}$ with $\tilde{\Gamma} \prec \Gamma$ are to the left of the dotted line and below the dashed line.

β : Points $\tilde{\Gamma}$ with $\Gamma \prec \tilde{\Gamma}$ are to the right of the dotted line and above the dashed line.

γ : Points $\tilde{\Gamma}$ incomparable with Γ .

After this it is clear from Eqs. (4.2) and (4.6) that $\mu_{KW} \geq \tilde{\mu}_{KW}$ is just a necessary but not sufficient condition for $\tilde{\Gamma} \prec \Gamma$. On the other hand, both μ_{TSF} and μ_g depend just on the distance of the point Γ to the origin. Some simple geometry shows that within the region $\mu_S \geq \mu_I$ the circle passing through Γ lies always on the γ sectors, so that μ_{TSF} and μ_g provide necessary but not sufficient conditions regarding majorization.

A simple example of incomparable states is provided by $\mu_S = 1$, $\mu_I = 0$ and $1 \geq \tilde{\mu}_S = \tilde{\mu}_I \geq 1/2$. In this case we have always $\tilde{\mu}_{KW} \geq \mu_{KW}$. In Fig. 3 we have represented $R_2(\tilde{\Gamma}) - R_2(\Gamma)$ and $R_1(\tilde{\Gamma}) - R_1(\Gamma)$. It can be appreciated that for $0.78 > \tilde{\mu}_S > 0.71$ the two Rényi entropies provide contradictory conclusions: this is $R_1(\tilde{\Gamma}) > R_1(\Gamma)$ versus $R_2(\tilde{\Gamma}) < R_2(\Gamma)$. We have plotted as well $\tilde{\mu}_{TSF}^2 - \mu_{TSF}^2$ around this same region showing regions where this measure of interferometric coherence contradicts the entropies. Moreover, in Fig. 4 we plot the ordered partial sums $S_n(\Gamma)$ and $S_n(\tilde{\Gamma})$ for $\tilde{\mu}_S = \tilde{\mu}_I = 0.75$ showing the lack of majorization.

V. FULLY POLARIZED VERSUS PARTIALLY POLARIZED BEAMS

Let us combine a fully polarized beam linearly polarized along then axis x with a partially polarized beam with

$$\begin{aligned} \Gamma_{1,1} &= I \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \Gamma_{2,2} &= I \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \\ \Gamma_{1,2} &= I \begin{pmatrix} \mu_S & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (5.1)$$

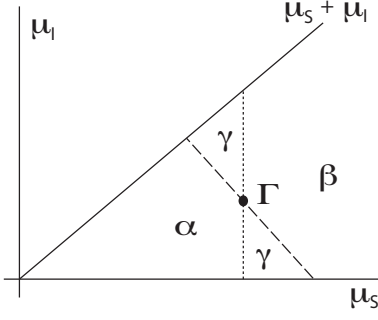


FIG. 2: Graphical translation of relations (4.6). The dotted line represents the condition $\mu_S \geq \tilde{\mu}_S$, while the dashed line represents the second condition $\mu_S + \mu_I \geq \tilde{\mu}_S + \tilde{\mu}_I$.

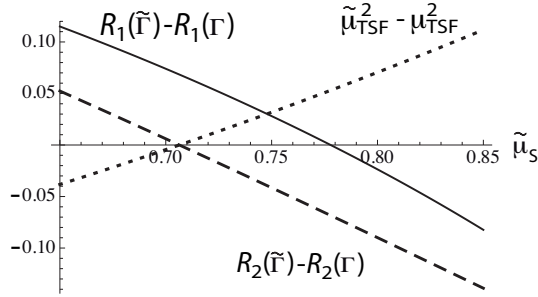


FIG. 3: Plot of $R_1(\tilde{\Gamma}) - R_1(\Gamma)$ (solid) and $R_2(\tilde{\Gamma}) - R_2(\Gamma)$ (dashed), and $\tilde{\mu}_{TSF}^2 - \mu_{TSF}^2$ (dotted), as functions of $\tilde{\mu}_S$ for $\mu_S = 1$, $\mu_I = 0$ and $\tilde{\mu}_S = \tilde{\mu}_I$.

where we will assume $\delta \geq 0$, and I represents the intensity of each component that will play no role on the following.

The corresponding values of the measures of polarization and interferometric coherence are:

$$P_1 = 1, \quad P_2 = \frac{|1 - \delta|}{1 + \delta}, \quad (5.2)$$

$$\mu_{TSF}^2 = \mu_{KW}^2 = \frac{\mu_S^2}{1 + \delta}, \quad (5.3)$$

and

$$\mu_g^2 = \frac{4 + 8\mu_S^2 + 3\delta^2 - 4\delta}{3(2 + \delta)^2}. \quad (5.4)$$

The dependence on μ_S is the expected one: larger μ_S implies both larger interferometric coherence and larger global coherence. On the other hand, the dependence on δ is more complicated. The degree of polarization has a minimum at $\delta = 1$, while interferometric coherence always decreases when increasing δ . Regarding global coherence we have that for fixed μ_S it has a minimum at $\delta = 1 + \mu_S^2$. Thus this example provides an interesting

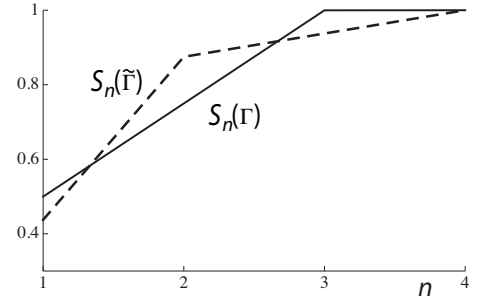


FIG. 4: Plot of the ordered partial sums $S_n(\Gamma)$ (solid) and $S_n(\tilde{\Gamma})$ (dashed) for two unpolarized waves with $\mu_S = 1$, $\mu_I = 0$, and $\tilde{\mu}_S = \tilde{\mu}_I = 0.75$, showing the lack of majorization. Points have been joined by continuous lines as an aid to the eye.

competition between polarization and interferometric coherence. From now on we will focus on the dependence on δ for fixed μ_S .

In this case, since there is always a vanishing eigenvalue we have $S_3 = S_4 = 1$ and majorization is just determined by the first two partial sums in Eq. (3.3). Taking into account that $S_1 = \lambda_1^\downarrow$ and $S_3 = S_2 + \lambda_3^\downarrow = 1$ we get a very simple relation already found in the three-dimensional problem considered in Ref. [13]. This is that $\tilde{\Gamma} \prec \Gamma$ is equivalent to

$$\lambda_1^\downarrow \geq \tilde{\lambda}_1^\downarrow, \quad \lambda_3^\downarrow \leq \tilde{\lambda}_3^\downarrow. \quad (5.5)$$

Regarding the ordering of the eigenvalues λ , we can distinguish three cases depending on the relation between δ and μ_S . The first one we consider is $\delta < 1 - \mu_S$, where the arrangement of λ in decreasing order is

$$\begin{aligned} \lambda_1^\downarrow &= \frac{1 + \mu_S}{2 + \delta}, & \lambda_2^\downarrow &= \frac{1 - \mu_S}{2 + \delta}, \\ \lambda_3^\downarrow &= \frac{\delta}{2 + \delta}, & \lambda_4^\downarrow &= 0. \end{aligned} \quad (5.6)$$

After Eqs. (5.5) and (5.6) for $\mu_S = \tilde{\mu}_S$ we readily get

$$\tilde{\Gamma} \prec \Gamma \Leftrightarrow \tilde{\delta} \geq \delta. \quad (5.7)$$

This is a quite expected result since for $\delta < 1$ increasing δ means both lesser degree of polarization and lesser interferometric coherence.

The opposite situation holds for $\delta > 1 + \mu_S$. In this case we will have $\delta > 1$ so we may expect that global coherence should emerge of a suitable balance between the opposed behaviors displayed by the degree of polarization and the interferometric coherence. The ordering of eigenvalues is

$$\begin{aligned} \lambda_1^\downarrow &= \frac{\delta}{2 + \delta}, & \lambda_2^\downarrow &= \frac{1 + \mu_S}{2 + \delta}, \\ \lambda_3^\downarrow &= \frac{1 - \mu_S}{2 + \delta}, & \lambda_4^\downarrow &= 0. \end{aligned} \quad (5.8)$$

After Eqs. (5.5) and (5.8) for $\mu_S = \tilde{\mu}_S$ we readily get

$$\tilde{\Gamma} \prec \Gamma \leftrightarrow \tilde{\delta} \leq \delta. \quad (5.9)$$

Roughly speaking, when δ increases the purity of the state provided by polarization overwhelms the decrease of the interferometric coherence.

Finally, in the intermediate situation $1 + \mu_S > \delta > 1 - \mu_S$ and $\mu_S = \tilde{\mu}_S$ the states are always incomparable unless $\tilde{\delta} = \delta$. This is because

$$\begin{aligned} \lambda_1^\downarrow &= \frac{1 + \mu_S}{2 + \delta}, & \lambda_2^\downarrow &= \frac{\delta}{2 + \delta}, \\ \lambda_3^\downarrow &= \frac{1 - \mu_S}{2 + \delta}, & \lambda_4^\downarrow &= 0, \end{aligned} \quad (5.10)$$

so that the two conditions in Eq. (5.5) are incompatible

$$\lambda_1^\downarrow \geq \tilde{\lambda}_1^\downarrow \leftrightarrow \tilde{\delta} \geq \delta, \quad \lambda_3^\downarrow \leq \tilde{\lambda}_3^\downarrow \leftrightarrow \tilde{\delta} \leq \delta. \quad (5.11)$$

VI. CONCLUSIONS

We have shown that majorization provides a powerful approach to the coherence conveyed by partially polar-

ized transversal waves. This is because it can be regarded as a kind of meta-measure of global coherence whose conclusions are respected by entropic measures of polarization and coherence. Moreover, majorization allows us to draw many parallels with coherence in quantum physics.

We have illustrated the approach by means of some simple but meaningful examples. The results are contrasted to other measures of polarization and interferometric coherence for vectorial waves. The situation is particularly interesting when polarization and interference behave in opposite ways.

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